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The evolution of trans-Alfvénic shocks in gases of finite electrical conductivity

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The stability of steady, plane, one-dimensional, trans-Alfvénic shocks to small normal disturbances (i.e. those in which the perturbed quantities are functions only of time and the distance from the plane of the shock wave) is discussed. The magnetic diffusivity of the ambient gas is taken to be very much greater than each of the viscous diffusivities and the thermal diffusivity.

It is confirmed in detail that all plane-polarized trans-Alfvénic shocks, except the 2–3 type (which is the only type that has no steady-state structure), are unstable to disturbances in those components of the magnetic field and velocity which are transverse to the plane of polarization. An incident Alfvén wave, consisting of a weak, diffusing current-sheet would initially cause the shock profiles of these transverse quantities to grow linearly with time, while outside this shock region steady, uniform states would be reached. An integral condition is obtained which, together with the relevant boundary conditions, determines the asymptotic shock profiles of the transverse quantities whenever the disturbance is such that a steady state is reached. This removes the puzzling arbitrariness of these profiles.

It is also shown that the '1-4' trans-Alfvénic shock is unstable to magneto-acoustic waves and contact fronts. A qualitative description of how it may be broken up is given. If the disturbance is of finite extent, a steady state is reached. An integral equation is obtained which, together with the relevant boundary conditions, determines the asymptotic steady-state shock-profiles for this case. This removes the apparent arbitrariness of these profiles.

The behaviour of '2–3' trans-Alfvénic shocks and of switch-on and switch-off shocks is discussed.

1. Introduction

A great deal of work has been done on the steady-state shock-structure of plane, oblique, magneto-gasdynamic shocks in a finite conductor (see, for example, Ludford 1959). The stability of these shocks to normal disturbances has been considered by Akhiezer, Liubarskii & Polovin (1958) and others, the ambient gas being treated as perfectly conducting. The following table includes

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Upstream longitudinal velocity, u_1 \ Downstream longitudinal velocity, u_2	4 ($u_2 < C_s$)	3 (C_s – α)	2 (α – C_f)
1 ($u_1 > C_f$)	MAGNETO-ACOUSTIC Waves, 7 in and 3 out (too few out) Shock structure, not unique *UNSTABLE* ALFVÉN Waves, 3 in and 1 out (too few out) Shock structure, not unique even if the shock profiles of the magneto-acoustic quantities are known explicitly	MAGNETO-ACOUSTIC Waves, 6 in and 4 out (correct number out) Shock structure, unique *UNSTABLE* ALFVÉN Waves, 3 in and 1 out (too few out) Shock structure, not unique	MAGNETO-ACOUSTIC Waves, 6 in and 4 out (correct number out) Shock structure, unique *STABLE* ALFVÉN Waves, 2 in and 2 out (correct number out) Shock structure, unique (i.e. $V_z = B_z = 0$)
2 (α – C_f)	MAGNETO-ACOUSTIC Waves, 6 in and 4 out (correct number out) Shock structure, unique *UNSTABLE* ALFVÉN Waves, 3 in and 1 out (too few out) Shock structure, not unique	MAGNETO-ACOUSTIC Waves, 5 in and 5 out (too many out) Shock structure, non-existent *NON-STATIONARY* ALFVÉN Waves, 3 in and 1 out (too few out) (Steady-state equations cannot be examined because V_x, etc., have no steady-state shock-structure)	
3 (C_s – α)	MAGNETO-ACOUSTIC Waves, 6 in and 4 out (correct number out) Shock structure, unique *STABLE* ALFVÉN Waves, 2 in and 2 out (correct number out) Shock structure, unique (i.e. $V_z = B_z = 0$)		

TABLE 1. The properties of oblique magneto-gasdynamic shocks

a summary of this work. The comments on the structure of the transverse† quantities is made as a result of the work outlined in this paper. C_f and C_s are the velocities of propagation of magneto-acoustic waves ($C_f > C_s$). α, the Alfvén velocity, is the velocity of propagation of transverse disturbances. Contact fronts are convected with the fluid. The numbers along the side of the table refer to the end states. The classification is that given by Germain (1959) and used by Shercliff (1960). A shock of the '1–4' variety is subject to the remarks of the top left-hand box. Similarly, the comments in the box at the top right-hand corner of the table refer to a '1–2' shock.

In each box, a count of the ten magneto-acoustic waves and contact fronts (magneto-acoustic) coming into and going away from the shock is made, and a comment is added as to whether or not these are in the right balance, i.e. 6–4. The Alfvén waves (Alfvén), i.e. those waves involving the transverse quantities only, have been similarly treated.

Below each count of the waves, both magneto-acoustic and Alfvén, a note is made on the shock structure of the quantities involved in those waves, for the case in which the magnetic diffusivity is dominant. This is a situation often met in practice. In such a plasma, a shock consists of a region of ohmic dissipation within which a very much thinner subshock, which corresponds to an ordinary gas dynamic shock, may be obtained.‡ All the diffusivities, except the magnetic diffusivity, are effectively taken as zero outside these subshocks. Continuum, macroscopic theory is accurate outside subshocks.

The '2–3' shock has no steady-state structure and is consequently termed non-stationary.

The table indicates the connexion between the non-existence, uniqueness or non-uniqueness of the shock structure and the excess, correctness or deficiency of the number of outgoing waves.

In a previous paper (Todd 1964) the evolution of trans-Alfvénic, *normal* shocks was discussed in detail. The introduction given there will clarify the preceding remarks of this section and will give the reader a much more extensive introduction to the subject.

Some comment on the notation, etc., which is employed in this paper is given below. The word 'trans-Alfvénic' will be abbreviated to T.A. and the adjective 'trans-Alfvénic normal' will be shortened to T.A.N. The '1–3', '1–4' and '2–4' shocks are all trans-Alfvénic. The undisturbed picture is one of a T.A. shock at rest in a finite conductor. Our basic Cartesian axes, $OXYZ$, are chosen moving in the gasdynamic subshock§ (discontinuity) such that the y- and z-components of the electric field are zero, Oz being chosen parallel to the direction of net current flow in the shock region. This means that the z-components of magnetic field and velocity outside the shock region are zero, i.e. the magnetic field and

† For simplicity axes have been chosen, moving in the shock, such that both the magnetic field and velocity outside the shock region are contained in one plane. The normal to this plane is in the transverse direction.

‡ Whether or not a subshock exists depends on the details of the end states of the shock.

§ For those T.A. shocks not containing a gasdynamic subshock, axes are chosen 'moving in the shock'. The '1–4' T.A. shock always contains a subshock. The others may or may not contain one.

velocity, outside the shock region, are plane polarized. Oz points in the transverse direction. Ox points in the direction of variation and is normal to the shock. Relative to this set of axes, $\mathbf{V}$, the velocity vector, is parallel to $\mathbf{B}$, the magnetic field vector (see, for example, Shercliff 1960), outside the shock region.

The upstream region is referred to as region 1 and all quantities evaluated there are given the suffix 1. A similar statement holds for the downstream region: region 2.

2. The stability of trans-Alfvénic shocks to transverse disturbances

The disturbances considered are functions of x and t (time) only. The full title for such perturbations is 'normal disturbances in the transverse quantities'. However, we shall mostly use the term 'transverse disturbances'.

2.1. *Equations and boundary conditions*

Let us consider the evolution of the general T.A. shock, subject only to the density, magnetic diffusivity and the normal component of the velocity vector having a steady-state shock profile in the undisturbed state. This includes all but the '2–3' species of T.A. shock. The '2–3' shock will be the subject of discussion in a later section.

The profiles of the three aforementioned quantities are explicit functions of x for the '1–3' and '2–4' T.A. shocks. However, for the '1–4' T.A. shock the steady-state forms of V_x, etc., are not explicit functions of x but also involve (implicitly) an apparently indeterminate constant.

The equations governing small disturbances in the transverse quantities are (in M.K.S. units):

$$\frac{\partial B_z}{\partial t}+u\frac{\partial B_z}{\partial x}-\frac{\partial}{\partial x}\left(\lambda\frac{\partial B_z}{\partial x}\right)+B_z\frac{\partial u}{\partial x}-B_x\frac{\partial V_z}{\partial x}=0, \tag{1}$$

and

$$\rho\frac{\partial V_z}{\partial t}+\rho u\frac{\partial V_z}{\partial x}-\frac{B_x}{\mu}\frac{\partial B_z}{\partial x}=0, \tag{2}$$

where B_x, etc., are the Cartesian components of the magnetic-flux-density vector, $\mathbf{V}=(V_x,V_y,V_z)$ is the velocity vector, ρ the density, μ the magnetic permeability; λ the magnetic diffusivity is $1/(\mu\sigma)$, σ being the electrical conductivity. B_x is constant in value.

The boundary conditions for V_z and B_z across a gasdynamic discontinuity are

$$\left[\lambda\frac{\partial B_z}{\partial x}-uB_z+B_xV_z\right]=0, \tag{3}$$

where the square brackets are used to denote the change of the quantity enclosed. Equation (3) expresses the fact that the y-component of the electric field is continuous across a gasdynamic discontinuity

$$\left[V_z-\frac{B_x}{\mu\rho u}B_z\right]=0. \tag{4}$$

Equation (4) requires that the transverse momentum flux plus Maxwell stresses are in balance.

$$[B_z]=0. \tag{5}$$

This last condition is a consequence of the fact that no appreciable current sheet can flow within a shock of width very much less than that which is based on ohmic dissipation.

The three remaining boundary conditions involve only p, ρ, and V_x, and they are the well-known Rankine–Hugoniot equations. The gasdynamic subshocks have been observed in practice and are well behaved, i.e. even when time changes in the flow are taking place, the rate at which these subshocks absorb mass, momentum, energy and magnetic flux is negligible, and so the boundary conditions which have just been derived are valid across them even when we study unsteady phenomena.

In equations (1)–(5), V_x has been set equal to u, its steady-state form. ρ and λ have been similarly treated. Terms of second order have been neglected. Unless the shock is '1–4', this linearized form of the equations and boundary conditions is correct to the order of retainment of terms even if magneto-acoustic disturbances are taking place simultaneously. It is shown in §3 that the structure of a '1–4' shock could change significantly in this case. Hence, when directly applying the results of this section to a '1–4' case, we would be assuming that no appreciable change in the shock structure of V_x, etc., is taking place. Even if such a change took place, the results of this section and §3 can be combined (see §3.7) to obtain the correct result.

$$b = (B_z/B_x), \quad v = (V_z/u) \quad \text{and} \quad m = (V_x/\alpha) = +\sqrt{(\mu\rho V_x^2/B_x^2)}$$

are useful non-dimensional quantities. To our order of retainment of terms, only the steady-state form of m enters this problem.

We shall only consider those disturbances in which B_z and V_z tend to constant values as $|x| \to \infty$ and are such that

$$\int_0^{\pm\infty} \{b-(b)_{\pm\infty}\}\,dx \quad \text{and} \quad \int_0^{\pm\infty} \{v-(v)_{\pm\infty}\}\,dx$$

are all finite. One could not expect to find asymptotic solutions for disturbances which are not of this type. Let us define the following quantities

$$\Lambda(t) = \int_{-\infty}^{0} \{b-(b)_{-\infty}\}\,dx + \int_0^{+\infty} \{b-(b)_{+\infty}\}\,dx, \tag{6}$$

where Λ is a measure of the flux of B_z per unit strip in the (x, y)-plane, the strip being directed along Ox; and

$$\chi(t) = \int_{-\infty}^{0} \{v-(v)_{-\infty}\}\,dx + \int_0^{+\infty} \{v-(v)_{+\infty}\}\,dx, \tag{7}$$

where χ is directly proportional to the z-component of momentum per unit y, z.

From equations (1) and (3), it can easily be shown that

$$\frac{d\Lambda}{dt} = \left[\lambda\,\frac{\partial b}{\partial x} - ub + V_z\right]_{-\infty}^{+\infty}. \tag{8}$$

Equation (8) relates the rate of charge of magnetic flux to the mismatch of the upstream and downstream values of the electric field far from the shock region.

Similarly, it can be shown that

$$\underset{\substack{\text{Rate of increase}\\\text{of transverse}\\\text{momentum}}}{\rho\frac{d\chi}{dt}} = [\underset{\substack{\text{Lorentz force}\\(J\wedge B)}}{\rho u m^{-2} b} - \underset{\substack{\text{Momentum}\\\text{flux}}}{\rho V_z}]^{+\infty}_{-\infty}. \tag{9}$$

Obviously if the disturbance is of finite extent, i.e. $(B_z)_{\pm\infty}$ and $(V_z)_{\pm\infty}$ are zero, then Λ and χ are constant in value because the right-hand sides of equations (8) and (9) are identically zero. If the disturbance under consideration is such that $(B_z)_{\pm\infty}$ and $(V_z)_{\pm\infty}$ are not all zero, then Λ and χ will, in general, increase at a constant rate with time because the right-hand sides of equations (8) and (9) will have, in general, constant, non-zero values.

It is possible to obtain expressions similar to (8) and (9) even when no linearization is introduced. This is illustrated by equations (30) and (31) of § 3.

2.2. *The steady-state shock-structure*

Let us suppose that b and v tend to constant values at the edge of the shock which satisfy the appropriate boundary conditions, i.e. equations (3), with $\lambda(\partial B_z/\partial x) = 0$, and (4). Admittedly our Cartesian set of axes has been chosen such that in the undisturbed steady state b and v both tend to zero as $|x| \to \infty$. However, an initial disturbance might arise which would produce a steady-state situation in which b and v both have non-zero values at the edge of the shock layer. Hence the more general case is examined. The steady-state equations can be obtained from (1) and (2). They are

$$\lambda\frac{db}{dx} + u(m^{-2}-1)\,b = A_1, \tag{10}$$

and

$$V_z = um^{-2}b + A_2, \tag{11}$$

where

$$A_1 = \{ub(m^{-2}-1)\}_\infty = \text{const.} \quad \text{and} \quad A_2 = \{V_z - um^{-2}b\}_\infty = \text{const.}$$

The solution of equation (10) is

$$b = A\Phi(x) + A_1\int_0^x \lambda^{-1}\Phi^{-1}\,d\xi\,\Phi(x), \tag{12}$$

where

$$\Phi(x) = \exp\left\{\int_0^x \frac{u}{\lambda}(1-m^{-2})\,d\eta\right\},$$

and A is constant.

These results apply to any kind of oblique shock. It is obvious that A is indeterminate if, and only if, $(m)_{-\infty} > 1$ and $(m)_{+\infty} < 1$. This is simply the condition that the shock is trans-Alfvénic. Thus B_z and V_z have an indeterminate shock structure in all T.A. shocks, except the 2–3 one,† even when the end states are zero (i.e. $A_1 = 0 = A_2$). In such a case V_z and B_z can take small, but finite, values within the shock region.

† This is the species which has been omitted from discussion in this section.

2.3. *An initial disturbance of finite extent*

The author proposes that a small disturbance of finite extent will, in general, deposit some flux of b and v at the gasdynamic discontinuity, and that the balance is propagated downstream via the one diffusing Alvén wave which it can emit. It is this asymptotic state which is studied in this subsection. It is contended that the outgoing wave is of finite extent. Let the flux of b in the outgoing wave be F_1 (i.e. F_1 is the contribution to Λ from the outgoing wave). Now the length scale of this wave will increase with time until we virtually have the same relationship between b_2 and v_2 as for the infinitely conducting case. This relationship is $v_2 = (-m_2^{-1}b_2)$. Hence, it is asserted that the flux of v_2 in the outgoing wave is $-m_2^{-1}F_1$.

In a previous paper (Todd 1964), the reaction of a T.A.N. shock, to a disturbance of finite extent, was examined. The asymptotic form of the shock profile of B_z and V_z were obtained by two different methods, both of which gave the same answer. The first method is that used later in this subsection and depends on the aforementioned assumptions. The second method employed contour integrations and was independent of any of the assumptions made in the first paragraph of this subsection.

Now the flux of b contained in the shock region, F_2 (say), is given by

$$F_2 = A\int_{-\infty}^{+\infty}\Phi\,dx = AR_1 \quad \text{(say)}.$$

The corresponding flux of V_z is

$$A\int_{-\infty}^{+\infty} m^{-2}\Phi\,dx = AR_2 \quad \text{(say)}.$$

Strictly speaking the upper limit on these integral signs should be a point between the shock region and the outgoing wave. However, for all practical purposes this may be taken as being infinity. Now since Λ and χ are conserved, we require that

$$F_1 + AR_1 = \Lambda, \quad \text{and} \quad -m_2^{-1}F_1 + AR_2 = \chi.$$

Hence

$$A = (\Lambda + m_2\chi)/(R_1 + m_2 R_2). \tag{13}$$

This is the integral expression which, together with the relevant boundary conditions, fixes the shock structure of B_z and V_z when steady, null states exist on either side of the shock region. Now if A had a non-zero value, A_0 (say), at our initial time and if the fluxes of b and v in the disturbance were Λ_0 and χ_0, respectively, the right-hand side of (13) would reduce to

$$A_0 + (\Lambda_0 + \chi_0)/(R_1 + m_2 R_2).$$

The results of this section illustrate the manner in which T.A. shocks can collect flux. The constant 'A' of equation (13) is, of course, a generalization of what is obtained in the previous paper (Todd 1964) and reduces to that expression for the degenerate T.A.N. case.

2.4. *The complementary initial value problem*

At $t = 0$, $b(x < 0)$, $b(x > 0)$, $v(x < 0)$ and $v(x > 0)$ are constant in value which means that $b(x < 0) = b_1 =$ constant, etc. Let $b_1 = f_1$, $b_2 = f_2$, $v_1 = g_1$ and $v_2 = g_2$. The most general disturbance of the type discussed in § 2.1 can be split into one of the above kind and one of finite extent. Hence, when the solution to this complementary problem is found, the solution of the 'general' disturbance is known.

In general, Λ and χ rise linearly with time. In accordance with the result obtained in the degenerate T.A.N. case, an asymptotic solution, for large t, of the kind given below is sought for b, at, and near, the shock

$$b = \Psi(x) + \beta t \Phi(x), \tag{14}$$

where β is a dimensional constant. Equations (1) and (2) may be rewritten as

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(\lambda \frac{\partial}{\partial x}\right) + u\frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\right) b - \frac{\partial V_z}{\partial x} = 0, \tag{15}$$

and

$$u^2 m^{-2} \frac{\partial b}{\partial x} - \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) V_z = 0. \tag{16}$$

Hence

$$\left\{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \times u\right)\left(\frac{\partial}{\partial x}\left(\lambda\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \times u\right) + \frac{\partial}{\partial x}\left(u^2 m^{-2}\frac{\partial}{\partial x}\right)\right\} b = 0. \tag{17}$$

Let us call the operator on b in equation (17) $L(\partial/\partial t, \partial/\partial x)$. Let

$$L(0, d/dx) = L(D) = L \quad \text{and} \quad M \equiv \lambda D - u(1 - m^{-2}).$$

Then, using the fact that um^{-2} is constant, it can be shown that

$$L(D) \equiv \frac{d}{dx}\left(u\frac{d}{dx}\left\{\lambda\frac{d}{dx} - u(1 - m^{-2})\right\}\right) \equiv \frac{d}{dx}\left(u\frac{d}{dx}\{M(D)\}\right).$$

$\{L(D)\}b = 0$, is, of course, satisfied by $b = A\Phi(x)$. It is required that

$$\left\{L\left(\frac{\partial}{\partial t}, D\right)\right\}\Psi + \beta\left\{L\left(\frac{\partial}{\partial t}, D\right)\right\}(t\Phi(x)) = 0.$$

Thus

$$\{L(D)\}\Psi = \beta\frac{d}{dx}\{(1 + m^{-2})u\Phi\},$$

and therefore

$$\{M(D)\}\Psi = \beta\int_0^x (1 + m^{-2})\Phi\, d\xi + N_1. \tag{18}$$

N_1 is a constant. The other constant of integration must be identically zero. Then

$$\Psi(x) = A\Phi + \int_0^x \lambda^{-1}\left\{\beta\int_0^\xi (1 + m^{-2})\Phi\, d\eta + N_1\right\}\Phi^{-1} d\xi\, \Phi(x). \tag{19}$$

A is a constant. In the T.A.N. case, a diffusing step Alfvén wave was propagated away from the shock region. Let us assume that this is again so and that just downstream of the shock region $b \to (b_2 + C)$ and $v \to (v_2 - m_2^{-1}C)$. For the purposes of the above solution 'just downstream of the shock region' is the point $x = +\infty$. N_1 is easily evaluated from equation (19) as

$$N_1 = \{u(m^{-2} - 1)b\}_{\pm\infty} - \beta\int_0^{\pm\infty}(1 + m^{-2})\Phi\, d\xi. \tag{20}$$

Hence, since the same value of N_1 must be obtained whether we take $+\infty$ or $-\infty$ as our outer limit, it is required that

$$(R_1+R_2)\beta - u_2(m_2^{-2}-1)\,C = u_1(1-m_1^{-2})f_1 + u_2(m_2^{-2}-1)f_2. \tag{21}$$

(The definitions of R_1 and R_2 are given in § 2.3.) Let us now find V_z in the shock region. It follows from equation (15) that

$$V_z = \beta\int_{-\infty}^{x}\Phi(\xi)\,d\xi + ub - \lambda\frac{\partial b}{\partial x} - (ub)_{-\infty} + (V_z)_{-\infty}. \tag{22}$$

The boundary conditions on V_z are satisfied if

$$R_1\beta + u_2(1+m_2^{-1})\,C = (u_1 f_1 - u_2 f_2) - (u_1 g_1 - u_2 g_2). \tag{23}$$

The values of β and C are obtained by solving the simultaneous pair of equations (21) and (23). This gives

$$\beta K_2 = (u_2^2/\lambda_2)\,((2-m_2^{-1})f_1 + (m_2^{-1}-1)\{f_2 - m_2(zg_1 - g_2)\}), \tag{24}$$

where
$$K_2 = (u_2/\lambda_2)\,(R_1 + m_2 R_2), \tag{25}$$

and
$$m_2^{-1}(m_2^{-1}+1)\,K_2 C$$
$$= (u_2/\lambda_2)\,(\{(g_2 - zg_1) + m_2^{-2}(f_1 - f_2)\}R_1 + (g_2 - zg_1 + zf_1 - f_2)\,R_2). \tag{26}$$

For the degenerate T.A.N. case, (24) and (26) yield the results obtained in the previous paper (Todd 1964). In all this work the constant 'A' of equation (19)† has remained an indeterminate constant, i.e. the governing equations are satisfied whatever its value. This constant can presumably be obtained only by examining the initial value problem, just as was done by Todd (1964) for the degenerate T.A.N. case.

However, all the important features of the asymptotic state have been obtained. These are that the shock profiles of b and v grow linearly with time, while outside the shock region b and v take steady uniform, values. Non-linear effects must eventually dominate and disrupt the shock structure in some way.

2.5. *The general transverse disturbance of the type discussed in* § 2.1

In this case the asymptotic forms of $b(x,t)$ and $v(x,t)$ are found by treating the disturbance as a combination of one of the type considered in § 2.3 and one of the kind considered in § 2.4. The sums of the separate solutions give the answers for b and v in the general case.

If $\beta = 0$ then a steady state will be reached in which b and v tend to non-zero values outside the shock region. We note that, in such a case,

$$A = A(f_1, f_2, g_1, g_2)$$
$$+\left(\frac{u_2}{\lambda_2}\right)K_2^{-1}\left(\int_{-\infty}^{0}\{b - f_1 + m_2(v - g_1)\}\,dx + \int_{0}^{+\infty}\{b - f_2 + m_2(v - g_2)\}\,dx\right), \tag{27}$$

where $b = b(x,0)$ and $v = v(x,0)$. This integral equation, together with the relevant boundary conditions, fixes the change in the shock profiles of b and V_z due to the passage of such a disturbance. Since we cannot evaluate $A(f_1, f_2, g_1, g_2)$ analytically, this last statement is rather academic.

† We shall refer to this constant as $A(f_1, f_2, g_1, g_2)$.

2.6. *The effect of shear viscosity*

It is interesting to consider what happens to the steady-state structure of B_z and V_z when the other diffusivities are taken into account. We shall only consider those T.A. shocks within which V_x, λ and the shear viscosity η have a steady-state structure. Which T.A. shocks this excludes, depends on the ordering of the diffusivities. It is not intended to go into this question in detail. The reader is referred to Germain (1960), Kulikovskii & Liubimov (1961) and Anderson (1963). However, it is in most cases a rather academic exercise since very few of the possible orderings of the diffusivities can be obtained in practice. The only one which would enter the equation explicitly is that corresponding to the shear viscosity. The steady-state linearized equations are now

$$[\lambda(d/dx)-u]\,b+V_z = A_1, \tag{28}$$

$$um^{-2}b+\left(\frac{\eta}{\rho u}\frac{d}{dx}-1\right)V_z = A_2. \tag{29}$$

A_1 and A_2 are constants. It follows that

$$\left(\nu\frac{d}{dx}-u\right)\left(\lambda\frac{d}{dx}-u\right)b-u^2m^{-2}b = -u(A_1+A_2), \tag{30}$$

where $\nu = \eta/\rho$. Let us suppose that $\lambda(x)$, $\nu(x)$, $u(x)$ and $m(x)$ exist and are known. Examination of the solution of (30) at the edge of the shock layer will reveal the uniqueness or otherwise of the profile of b.

At the edge of the shock b takes the form

$$b = A\,e^{\alpha_+x}+B\,e^{\alpha_-x}+(\text{particular integral}),$$

where $$\alpha_\pm = \tfrac{1}{2}u((\nu^{-1}+\lambda^{-1})\pm\surd\{(\nu^{-1}-\lambda^{-1})^2+4\nu^{-1}\lambda^{-1}m^{-2}\}).$$

If $m < 1, \alpha_+ > 0 > \alpha_-$. If $m > 1, \alpha_+ > \alpha_- > 0$. Hence the profile of b is given by

$$b = (\text{particular integral})+Af_1(x),$$

where $f_1\,e^{-\alpha_-x}\rightarrow 1$ as $x\rightarrow+\infty$. At the upstream edge

$$b\rightarrow(\text{particular integral})+A(C_1\,e^{\alpha_+x}+C_2\,e^{\alpha_-x}),$$

where C_1 and C_2 are numerical constants which could be computed in any given case. Hence B_z and V_z have arbitrary structure in those T.A. shocks within which λ, ν, u, B_y and V_y have a steady-state structure. The work of § 6 has been done already by Dr J. A. Shercliff in unpublished work and probably by other people too. However, the result does not seem to be widely known.

3. The stability of the '1–4' trans-Alfvénic shock to magneto-acoustic waves and contact fronts

The disturbances considered are functions of x and t only, i.e. they are normal disturbances.

3.1. *The equations and boundary conditions*

The undisturbed picture is one of a '1–4' T.A. shock at rest. The hydrodynamic discontinuity contained within this shock region is situated at $x = 0$.

Magneto-acoustic waves perturb p, ρ, etc., and thus cause a first-order change in the shock structure of these quantities. Allowance must therefore be made for a small movement of the shock front relative to its undisturbed motion. Let the displacement of the discontinuity from its undisturbed position be ϵ. New co-ordinates x', t are chosen where $x' = (x-\epsilon)$. Thus

$$\left(\frac{\partial A}{\partial x}\right)_t = \left(\frac{\partial A}{\partial x'}\right)_t \quad \text{and} \quad \left(\frac{\partial A}{\partial t}\right)_t = \left(\frac{\partial A}{\partial t}\right)_{x'} - \dot{\epsilon}\left(\frac{\partial A}{\partial x'}\right)_t,$$

where the 'dot' denotes differentiation with respect to time.

We shall now write out the governing equations in terms of x' and t, but we shall drop the dash in doing so. *The electric field* $\mathbf{E}$ *and* V *are still measured relative to the undisturbed system of axes.*

$$\rho\frac{\partial V_x}{\partial t} + \epsilon(V_x-\dot{\epsilon})\frac{\partial V_x}{\partial x} + \frac{\partial p}{\partial x} + \frac{B_y}{\mu}\frac{\partial B_y}{\partial x} = 0, \tag{31}$$

$$\frac{\partial \rho}{\partial t} + (V_x-\dot{\epsilon})\frac{\partial \rho}{\partial x} + \rho\frac{\partial V_x}{\partial x} = 0, \tag{32}$$

$$\rho\frac{\partial}{\partial t}\left(\rho+\frac{v^2}{2}\right) + \frac{\partial}{\partial t}\left(\frac{B_y^2}{2\mu}\right) + \rho(V_x-\dot{\epsilon})\frac{\partial}{\partial x}\left(e+\frac{v^2}{2}\right) + \frac{\partial}{\partial x}(pV_x) - \frac{\partial}{\partial x}(E_z B_y) = 0, \tag{33}$$

$$E_z = V_y B_x - B_y V_x + \lambda\frac{\partial B_y}{\partial x}, \tag{34}$$

$$\frac{\partial B_y}{\partial t} + (V_x-\dot{\epsilon})\frac{\partial B_y}{\partial x} - \frac{\partial}{\partial x}\left(\lambda\frac{\partial B_y}{\partial x}\right) - B_x\frac{\partial V_y}{\partial x} + B_y\frac{\partial V_x}{\partial x} = 0, \tag{35}$$

$$\rho\frac{\partial V_y}{\partial t} + \rho(V_x-\dot{\epsilon})\frac{\partial V_y}{\partial x} - \frac{B_x}{\mu}\frac{\partial B_y}{\partial x} = 0. \tag{36}$$

The boundary conditions are that at $x = 0$, i.e. the hydrodynamic discontinuity,

$$[\rho(V_x-\dot{\epsilon})] = 0, \tag{37}$$

$$[\rho(V_x-\dot{\epsilon})V_x + p] = 0, \tag{38}$$

$$[\rho(V_x-\dot{\epsilon})(e+\tfrac{1}{2}V^2) + pV_x] = 0, \tag{39}$$

$$\left[\lambda\frac{\partial B_y}{\partial x} - V_x B_y\right] = 0, \tag{40}$$

$$[V_y] = 0, \quad \text{and} \quad [B_y] = 0, \tag{41}$$

where e is the internal energy per unit mass, and E_z, etc., are the Cartesian components of the electric-field vector. Provided the transverse disturbances are small, the terms involving B_z and V_z which have been omitted from the above are of *second* order.

3.2. *Shock structure*

Let us examine the steady-state structure of a '1–4' T.A. shock. Within the region of a magneto-gasdynamic shock where ohmic diffusion only counts, it is required that

$$F_x = p + GV_x + B_y^2/(2\mu) = \text{const.}, \tag{42}$$

$$G = \rho V_x = \text{const.}, \tag{43}$$

$$H = h + \tfrac{1}{2}(V_x^2 + V_y^2) - E_z B_y/(\mu G) = \text{const.}, \tag{44}$$

$$E_z = V_y B_x - V_x B_y + \lambda(dB_y/dx) = 0, \tag{45}$$

and

$$F_y = GV_y - B_x B_y/\mu = \text{const.} \tag{46}$$

Even when the thermodynamics of the gas is known the problem is a fairly difficult two-dimensional one. There exists an infinity of possible shock profiles for the quantities concerned (see, for example, Ludford 1959). As has already been mentioned, each involves a 'subshock', i.e. a gasdynamic discontinuity. For the purpose of this paper, it is necessary to be able to pick out some quantity which, in addition to equations (12) to (16), will fix the shock structure. A suitable choice is $(B_y/B_x)_{x=0} = B^0$. The structure of each quantity in this shock is plotted as a function of B^0 and of x by starting with a given value of B^0, and computing outwards. In this way, a two-dimensional plot of quantities such as $B_y = B_y(x, B^0)$ (say), etc., would be obtained.

3.3. *The integral equations*

Let us define

$$I_\eta = \int_{-\infty}^{0} \{\eta - (\eta)_{-\infty}\}\,dx + \int_{0}^{+\infty} \{\eta - (\eta)_{+\infty}\}\,dx;\ \eta = G, F_x, B_y, \text{etc.}$$

Equation (1) may be rewritten as

$$\frac{\partial}{\partial t}(\rho V_x) + \frac{\partial}{\partial x}\{\rho(V_x - \dot{\epsilon})V_x + p + B_y^2/(2\mu)\} = 0.$$

Hence

$$\frac{dI_G}{dt} = \dot{I}_G = [G\dot{\epsilon} - F_x]_{-\infty}^{+\infty}, \tag{47}$$

where F_x and G are defined by equations (12) and (13), respectively. Similarly, it can be shown that

$$\dot{I}_\rho = [\rho\dot{\epsilon} - G]_{-\infty}^{+\infty}, \tag{48}$$

$$\dot{I}_{\mathscr{E}} = [\rho(e + \tfrac{1}{2}V^2)\dot{\epsilon} - GH]_{-\infty}^{+\infty}, \tag{49}$$

where H is defined by equation (14), and

$$\mathscr{E} = \rho(e + \tfrac{1}{2}V^2) + B_y^2/(2\mu).$$

Also

$$\dot{I}_{\rho V_y} = [\rho V_y \dot{\epsilon} - F_y]_{-\infty}^{+\infty}, \tag{50}$$

and

$$\dot{I}_{B_y} = [B_y\dot{\epsilon} + E_z]_{-\infty}^{+\infty}. \tag{51}$$

Let us eliminate $\dot{\epsilon}$ between the equations (47)–(51) and obtain four integral equations depending only on conditions at 'infinity'. Substituting from equation (18) into (47), (49), (50) and (51) we obtain

$$[\rho]\dot{I}_G - [G]\dot{I}_\rho = [G]^2 - [\rho][F_x], \tag{52}$$

$$[\rho]\dot{I}_{\mathscr{E}} - [\rho(e + \tfrac{1}{2}V^2)]\dot{I}_\rho = [\rho(e + \tfrac{1}{2}V^2)][G] - [\rho][GH], \tag{53}$$

$$[\rho]\dot{I}_{\rho V_y} - [\rho V_y]\dot{I}_\rho = [\rho V_y][G] - [\rho][F_y], \tag{54}$$

and

$$[\rho]\dot{I}_{B_y} - [B_y]\dot{I}_\rho = [B_y][G] + [\rho][E_z], \tag{55}$$

where $[A] \equiv [A]_{-\infty}^{+\infty}$. The integral equations derived in this section are valid even if the perturbed quantities are large.

3.4. *Fluxes in the outgoing waves*

In the next part of this section we will consider the reaction of the '1–4' T.A. shock to a small disturbance (in the magneto-acoustic quantities) of finite extent. Before doing this, it is necessary to consider the relationships between the fluxes of the various quantities in each of the three outgoing waves, these waves being of finite extent. Arguments analogous to those given at the beginning of § 2.3 show that these relationships are the same as those for cases in which R_m is infinite, i.e. those cases in which the length scale of the disturbance is very much larger than the natural length scale, (λ/u).

(*a*) *The contact front.* The gas has a change in density in this front. The quantities p, $\mathbf{V}$, and $\mathbf{B}$ are constant throughout the front. Let us call the flux of ρ in the wave F_0. This quantity consists of the integral, over the extent of the front, of the difference between ρ and its undisturbed value. The corresponding flux of $\mathscr{E}$ is $K_0 F_0$, where K_0 is a constant which is known explicitly if the thermodynamic properties of the ambient gas are specified.

(*b*) *The fast and slow magneto-acoustic waves.* If the mass flow through the shock is directed from left to right, both of these waves must be right travelling relative to the fluid. The relative change of each quantity within the wave is small compared to unity. The velocities of the waves are C_f and C_s, respectively. C_f and C_s are the positive roots of

$$C^4 - (a_2^2 + \alpha_2^2 + (\mu\rho_2)^{-1} B_{y_2}^2) C^2 + a_2^2 \alpha_2^2 = 0,$$

and $C_f > C_s$. $a_2^2 = ((\partial p/\partial\rho)_s)_2$.

From now on the suffixes f and s will be omitted from all quantities, and we shall simply discuss fluxes in a magneto-acoustic wave. The suffix f or s can be substituted in the appropriate places when it is desired to refer only to the fast or the slow wave.

Let the flux of B_y in the wave be F. Then the flux of ρ in this wave is

$$\rho_2 \frac{C^2 - \alpha_2^2 - B_{y_2}^2/(\mu\rho_2)}{a_2^2 B_{y_2}} F.$$

The flux of V_y is $-B_x F/(\mu\rho_2 C)$, and the flux of ρV_x is

$$\rho_2 \frac{(C+u_2)\{C^2 - \alpha_2^2 - B_y^2/(\mu\rho_2)\}}{a_2^2 B_{y_2}} F.$$

The flux of ρV_y is

$$\left(\rho_2 \frac{\{C^2 - \alpha_2^2 - B_y^2/(\mu\rho)\}}{a_2^2 B_{y_2}} V_{y_2} - \frac{B_x}{\mu C}\right) F.$$

The flux of $\mathscr{E}$ in the wave can be written as KF. K would be known explicitly if the thermodynamics of the ambient gas was specified.

3.5. *A disturbance of finite extent*

Let B_0^0 be the value of B^0 at $x = 0$ in the undisturbed steady state and $K_4(B_0^0)$ be the contribution to $([\rho] I_{B_y} - [B_y] I_\rho)$ at $t = 0$ from the initial shock profiles of B_y and ρ. It is convenient to define I_4 as $([\rho] I_{B_y} - [B_y] I_\rho) - K_4(B_0^0)$ at $t = 0$. It follows from equation (55) that for a disturbance of finite extent $([\rho] I_{B_y} - [B_y] I_\rho)$ is constant. Again it is suggested that the disturbance will deposit some flux at the shock, the balance being propagated downstream via the two outgoing waves and the outgoing contact front. Let the contribution to $I_4 + K_4(B_0^0)$ from the new shock profiles be $K_4(B_N^0)$. B_N^0 is the new value of (B_y/B_x) at the gasdynamic discontinuity. The corresponding two expressions for equations (52), (53) and (24) are I_1 and K_1, I_2 and K_2, and I_3 and K_3, respectively. K_1, K_2, K_3 and K_4 will be known implicitly as functions of B_0. It is required that

$$K_1(B_N^0) - \rho_1[V_x]F_0 + [\rho]\,(\sum_{f,s} (C+u_2)F^\rho) = I_1 + K_1(B_0^0), \tag{56}$$

where
$$F^\rho = \rho_2 \frac{C^2 - a_2^2 - B_{y_2}^2/(\mu\rho_2)}{a_2^2 B_{y_2}} F,$$

$$K_2(B_N^0) + ([\rho]K_0 - [\rho(e+\tfrac{1}{2}V^2)])F_0 + \sum_{f,s} ([\rho]KF - [\rho(e+\tfrac{1}{2}V^2)]F^\rho) = I_2 + K_2(B_0^0), \tag{57}$$

$$K_3(B_N^0) - \rho_1[V_y]F_0 - \sum_{f,s}\left(\rho[V_y]F^\rho + [\rho]\frac{B_x}{\mu C}F\right) = I_3 + K_3(B_0^0), \tag{58}$$

and lastly
$$K_4(B_N^0) - [B_y]F_0 + \sum_{f,s} ([\rho]F - [B_y]F^\rho) = I_4 + K_4(B_0^0). \tag{59}$$

B_N^0 can thus be found implicitly in terms of the flux in the disturbance and its initial value. Even though all the perturbed quantities in the outgoing waves must remain small, there is no restriction on the change in B^0 brought about by the disturbance. The restriction does, however, mean that $\dot{\epsilon}$ remains small compared with the longitudinal velocity of the gas relative to the shock.

There still remains the mathematical possibility that the new value of B^0 is not unique. Let us suppose that B_N^0 is multivalued. If the value of one or more of the K's changes with these values of B_N^0, the flux in the outgoing waves would be changed. But this is impossible. Thus all of the K's must have the same values for the various possible values of B_N. This would mean that the shock can take on a completely new structure without consulting the outside world, a situation which cannot be tolerated. Thus it is concluded that the new value of B_N must be unique.

3.6. *The complementary disturbance*

This is that type of disturbance in which the perturbed quantities have constant, uniform values on each side of the gasdynamic discontinuity. In this case $([\rho]I_G - [G]I_\rho)$, etc., rise linearly with time and obviously the fluxes of the various quantities will be divided in some manner between the outgoing waves and the shock. However, as the shock profiles do not depend linearly on B_0, a solution in which the perturbed quantities have constant values just outside an unsteady shock region cannot be considered. Thus a solution of the type obtained in § 2.4 is not available. The process outlined above, whereby flux is continually being stored up in the shock region cannot go on indefinitely. At some stage the creation of entropy, etc., inside the shock is going to produce new shocks by choking and, or, splitting processes. Presumably one would then have a stable configuration of simple waves, contact fronts and shocks. There are, of course, certain disturbances in the above category which do not break up the '1–4' T.A. shock. These perturbations are such that the initial mismatches of $(E_z)_\infty$, etc., across the magneto-gasdynamic shock can be rectified by the emission of the three outgoing waves.

3.7. *The coupling between transverse and magneto-acoustic disturbances*

Let us consider a small disturbance, in B_y, V_y, p, ρ, V_x, V_z and B_z, of finite extent which significantly alters† the value of B^0. The results of § 2.3 cannot be directly applied to this problem. However, in the manner described in § 3.3, we can show that

$$\dot{I}_{\rho V_z} = [\rho V_z \dot{e} - F_z]^{+\infty}_{-\infty}, \tag{60}$$

and

$$\dot{I}_{B_z} = [B_z \dot{e} - E_y]^{+\infty}_{-\infty}. \tag{61}$$

Thus for disturbances of finite extent the fluxes of B_z and V_z are conserved. The asymptotic form for B_z at, and near, the shock is obviously that obtained in § 2.3 except that the value of B^0 which is used in the calculation of R_1 and R_2 must be B^0_N, i.e. the value of B^0 after the passage of the disturbances in B_y, etc.

For a completely arbitrary small disturbance the value of $(B_z/B_x)_{x=0}$ will, in general, be constantly modified by the disturbances in the magneto-acoustic quantities as well as by the perturbations in B_z and V_z.

4. The stability of the '2–3' trans-Alfvénic shock

The '2–3' T.A. shock has no steady-state structure. Thus the situation is one of two uniform regions, connected by the generalized Rankine–Hugonist equations, separated by a region in which time-changes must occur. This latter region may contain one or more gasdynamic discontinuities. It is still possible to obtain integral equations like those in § 3.3, though more than one ϵ may be involved. Thus the fluxes of the quantities in the breakdown picture are subject to the conservation-type conditions. The '2–3' T.A. shock is in fact a wave-generator, whereas the other T.A. shocks are flux-collectors. In order to discuss the evolution of the '2–3' shock, it is necessary to take this property into account as well as considering the incoming waves. However, the natural time-changes taking place in the 'shock-region' will, in general, lead to a non-linear breakdown involving stable shocks, simple waves, etc.

5. The fate of switch-on and switch-off shocks in finite conductors

The switch-on shock separates '1–2' transitions from '1–3' ones. Similarly, the switch-off shock divides shocks of the '2–4' species from those in the '3–4' category.

Both these shocks have a unique‡ steady-state structure. Furthermore, switch-on and switch-off shocks are stable to disturbances in the magneto-acoustic quantities.

Let us now discuss the effect of small, normal disturbances in B_z and V_z upon switch-on and switch-off shocks. The conservation-type equations (8) and (9) of § 2 are valid.

For a disturbance of finite extent, the fluxes of B_z and V_z are conserved. The author suggests that some flux will be deposited at the shock and the balance

† The perturbations in B_z and V_z are such that $(B_z/B_x)_{x=0}$ is always small compared to unity.

‡ The steady-state structure of the transverse quantities is $V_z = 0$ and $B_z = 0$.

propagated downstream in the form of a diffusing, right-travelling, Alfvén wave of finite extent. Since convection and propagation are in strict balance on one side of the shock, diffusion is the dominant physical process in that region. Hence any flux which is collected at the shock will tend to diffuse into the Alfvénic region, i.e. the region in which $m = 1$, in order that the shock may regain its original structure, namely $V_z = 0$ and $B_z = 0$.

Obviously, disturbances exist which will deposit flux at these shocks at a greater rate than it can be diffused away.† In such cases non-linear effects must eventually dominate.

It is worth discussing the actual breakdown of switch-on and switch-off shocks. Let us subject a switch-on shock to a disturbance of the 'complementary' type (see § 2.4). The final breakdown picture would consist of a magneto-gasdynamic shock which is just '1–2' followed by an Alfvén simple wave which rotates the magnetic field from its new direction back to the y-direction.‡ There will be magneto-acoustic wavelets, etc., present as well. The Alfvén simple wave will diffuse with time, but, as the '1–2' shock moves steadily away from it, the breakdown picture is a valid one.

It must be inferred that, under the action of an entirely arbitrary, small disturbance, one will always have a shock which is nearly switch-on plus an associated tail in the downstream region: this tail being closely related to an Alfvén simple wave. Consequently it is suggested that switch-on shocks are stable. For analogous reasons it is suggested that switch-off shocks are stable.

It must be pointed out that these conclusions on the stability of switch-on and switch-off shocks are not valid if the shocks are 'weak', i.e. if they are nearly null switch-on and null switch-off shocks, respectively. This is because we have assumed in deriving the breakdown picture that the perturbations in B_z are small compared to both B_y and B_x. A similar statement holds for the perturbations in V_z.

Those T.A. shocks which are just '1–3' or just '2–4' have similar breakdown configurations to switch-on and switch-off shocks, respectively. Thus the gradual transition from the stable super-Alfvénic and sub-Alfvénic species of shocks to the unstable trans-Alfvénic species is clearly revealed.

Conclusions

The technique employed in this paper, that of considering the fluxes of the perturbed quantities, is possible because the governing equations for p, ρ, $\mathbf{V}$ and $\mathbf{B}$ can all be written in the form $\partial A_i/\partial t + \partial(B_{ij})/\partial x_j$, where the A_i and the B_{ij} are functions of the dependent variables. (For scalar equations, $i = 1$.)

It is confirmed in detail that trans-Alfvénic shocks are unstable to normal disturbances in the transverse quantities. It is also concluded that the '1–4' trans-Alfvénic shock is unstable to normal perturbations in the magneto-acoustic quantities and that the '2–3' trans-Alfvénic shock will disintegrate

† This has been demonstrated analytically and computationally for switch-on and switch-off shocks. This work is as yet unpublished.

‡ The small amplitude waves which are emitted will, in general, cause a slight change in this direction.

independently of the presence of small disturbances. It must be remembered that all trans-Alfvénic shocks, except the '2–3' one, are unstable to small disturbances other than normal ones. Thus for a completely unrestricted small disturbance the instabilities detailed in this paper may be only a part of a very much more complex disintegration pattern.

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